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Hydrodynamic Equations for Attractive Particle Systems on $\mathbb{Z}$

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Hydrodynamic properties for a class of nondiffusive particle systems are investigated. The method allows one to study local equilibria for a class of asymmetric zero-range processes, and applies as well to other models, such as asymmetric simple exclusion and "misanthropes." Attractiveness is an essential ingredient. The hydrodynamic equations present shock wave phenomena. Preservation of local equilibrium is proven to hold away from the shocks. The problem of breakdown of local ergodicity at the shocks, which was investigated by D. Wick in a particular model, remains open in this more general setup.

KEY WORDS: Infinite-particle system; attractiveness; coupling; hydrodynamic equation; zero-range process; conservation law.

1. INTRODUCTION

In this article we investigate the hydrodynamic behavior of a class of stochastic particle systems under Euler scaling. More precisely, we are concerned with nondiffusive types of one-dimensional models, whose hydrodynamic equation will be a nonlinear conservation law, thus exhibiting shock waves for some initial data. We extend results contained in Refs. 11, 12, 16, 3, and 4; and our main goal is to provide a more unified treatment, exploiting the "attractiveness" present in our models. For simplicity we shall state the results for a class of zero-range processes,[1] but it will be clear how the same ideas apply to asymmetric simple exclusion processes,[16,3] as well as the more general models introduced in Ref. 5.

For the physical motivations and general discussions of the problems under study we refer to Refs. 6 and 15 and references cited therein.

As noticed by Morrey[14] in the study of fluid dynamics, the assumption of "preservation of local equilibrium" allows one to derive the

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Euler equation (hydrodynamic equation). This assumption involves a suitable family of initial measures and time and space scale changes (Euler scaling in his case). Nevertheless, very little is known concerning the validity of this assumption for realistic (physical) systems (see Ref. 15 and the references therein).

In the context of stochastic dynamics, which is simpler to deal with, there is a greater variety of examples, especially for diffusive-type scaling, where the hydrodynamic equation is a diffusion equation. In such situations there is a certain general theory, under the assumption of local equilibrium.[6]

Generally, however, checking preservation of local equilibrium is not easy, and the main point is that, for most of the examples so far, this already involves the identification of the parameters describing local equilibrium.[2]

In Refs. 3, 4, 16, and 18 examples have been studied which lead to a nonlinear conservation law, presenting shock waves. The connection between the behavior at the microscopic level and the propagation of shock waves deserves much study. One important result in this direction was obtained by D. Wick, who gave an example in which the local equilibrium assumption fails where shock waves occur. There is a rupture of "local ergodicity," leading to a superposition of different equilibria at the wavefront. (See also Refs. 2 and 7 for similar situations.)

Here we treat a more general class of models and show that in the smooth case, or away from the shock, we do have preservation of local equilibrium, the local parameters being described by the entropy solution of the hydrodynamic equation. The extremely interesting question concerning the behavior at the shock remains open in this more general setup.

In Section 2 we specify the class of models we will be dealing with, and state the main results. The proofs are given in Sections 3 and 4. Finally, in Section 5, we discuss their more general validity, including, e.g., models such as those in Ref. 5. The proofs given here do not use ergodic theorems or special theorems for queuing systems. These played an important role in Refs. 3, 4, and 18. We also obtain the results of Refs. 11 and 12 without resorting to the special stochastic order introduced in Ref. 11.

2. DESCRIPTION OF THE MODEL—RESULTS

In the one-dimensional zero-range process we have infinitely many particles moving on $\mathbb{Z}$; they are indistinguishable, so we keep track of the

[2] Recall also the possibility of deriving hydrodynamic equations without proving local equilibrium, and showing instead "propagation of chaos."[6,15]

occupation number at each $x \in \mathbb{Z}$, $\eta_t(x)$, which should be always finite. At each nonempty site x, the rate at which one of the particles will jump is a function of its occupation number $g(\eta_t(x))$, and it jumps according to a probability transition function $p(x, y)$.

The construction of a Markov process (η_t) on $\mathbb{N}^{\mathbb{Z}}$ according to this intuitive description in carried out in Refs. 8 and 1. Clearly, one must impose conditions on $g(\cdot)$ and $p(\cdot, \cdot)$ to avoid, e.g., infinitely many particles jumping to a fixed site. If $g(\cdot)$ is bounded and $p(\cdot, \cdot)$ is translation-invariant [i.e., $p(x, y) = p(0, y-x)$] with $\sum_y |y|\, p(0, y) < +\infty$, the construction can be done for any initial configuration $\eta \in \mathbb{N}^{\mathbb{Z}}$ (Ref. 8). Under more general conditions, such as relaxing the boundedness of g to $\sup_k |g(k+1) - g(k)| < +\infty$, it is necessary to restrict the set of allowed configurations to a suitable subset $E \subseteq \mathbb{N}^{\mathbb{Z}}$ (see Refs. 1 and 10 for such constructions).

For reasons that will be clear later, we make the following assumptions throughout Sections 2–4:

Assumptions 2.1.

(a) The function $k \to g(k)$ is monotone (nondecreasing) and bounded; $0 = g(0) < g(1)$.

(b) $p(x, y) = p(0, y-x)$ for all x, y integers; $\sum_y |y|\, p(0, y) < +\infty$; and

$$\gamma \overset{\text{def}}{=} \sum_y y p(0, y) \in (0, +\infty)$$

The previous description corresponds to the following pregenerator L, acting on cylinder functions f on $\mathbb{N}^{\mathbb{Z}}$:

$$Lf(\eta) = \sum_{x,\, y} g(\eta(x))\, p(x, y)[f(\eta^{x,y}) - f(\eta)] \tag{2.1a}$$

with

$$\eta^{x,y}(z) = \begin{cases} \eta(x) - 1 & \text{if} \quad z = x \\ \eta(y) + 1 & \text{if} \quad z = y \\ \eta(z) & \text{if} \quad z \neq x, y \end{cases} \tag{2.1b}$$

provided $\eta(x) \geqslant 1$, and $x \neq y$; $\eta^{x,y} \equiv \eta$ otherwise.

Under (a) and (b) we know (cf. Refs. 8, 10, and 1) that for each $\eta \in \mathbb{N}^{\mathbb{Z}}$ there exists a unique probability measure P_η on the Skorohod space[3] $D([0, +\infty); \mathbb{N}^{\mathbb{Z}})$ so that

[3] $\mathbb{N}^{\mathbb{Z}}$ is taken with product topology, given, e.g., by the metric

$$d(\eta, \xi) = \sum_x 2^{-|x|}(|\eta(x) - \xi(x)|)/[1 + |\eta(x) - \xi(x)|]$$

(i) $P_\eta[\eta(0)=\eta]=1$

(ii) Under P_η: $f(\eta_t)-\int_0^t Lf(\eta_s)\,ds$ is a martingale with respect to the canonical filtration, for each bounded cylinder function f.

Moreover, we do have:

(iii) P_η is Markovian with respect to the canonical filtration.

(iv) $\eta \to P_\eta(A)$ is Borel-measurable for each Borel set A in $D([0,+\infty),\mathbb{N}^{\mathbb{Z}})$.

We let $(S_t)_{t\geqslant 0}$ be the Markov semigroup associated to our process, i.e., $S_t f(\eta)=\int f(\eta_t)\,dP_\eta$ for f bounded continuous function on $\mathbb{N}^{\mathbb{Z}}$. If μ is a probability measure on $\mathbb{N}^{\mathbb{Z}}$, μS_t will denote the law of η_t when η_0 is distributed according to μ, i.e., $\mu S_t(f)=\int S_t f(\eta)\,\mu(d\eta)$. From the construction in Ref. 1 (or by several other methods) we know that (S_t) is a strongly continuous Markov (and Feller) semigroup of operators on the space of bounded continuous functions of $\mathbb{N}^{\mathbb{Z}}$, whose generator extends L.

In the final section we shall make a few comments on relaxing boundedness conditions on $g(\cdot)$ and other extensions. Nevertheless, the assumption of monotonicity of $g(\cdot)$ will be crucial in this article. Our arguments rely strongly on the "attractiveness" of the model, which is a consequence of such monotonicity. This notion of "attractiveness"[13] corresponds to a certain order preservation by the semigroup (S_t). For such a definition we consider on $\mathbb{N}^{\mathbb{Z}}$ the partial ordering given by

$$\eta \leqslant \xi \qquad \text{if} \quad \eta(x)\leqslant \xi(x) \quad \text{for all} \quad x \tag{2.2}$$

and the corresponding stochastic order between probability measures on $\mathbb{N}^{\mathbb{Z}}$: $\mu_1\leqslant\mu_2$ means that we can construct $\tilde{\mu}$ probability measure on $\mathbb{N}^{\mathbb{Z}}\times\mathbb{N}^{\mathbb{Z}}$ with $\tilde{\mu}(\cdot\times\mathbb{N}^{\mathbb{Z}})=\mu_1$, $\tilde{\mu}(\mathbb{N}^{\mathbb{Z}}\times\cdot)=\mu_2$, and supported by $\{(\eta,\xi):\eta\leqslant\xi\}$ (see Ref. 13 for equivalent definitions). It is well known[1] that the monotonicity of $g(\cdot)$ guarantees that $\mu_1\leqslant\mu_2$ implies $\mu_1 S_t\leqslant\mu_2 S_t$ for each t. In fact, for any given configurations $\eta\leqslant\xi$ it is possible to construct a coupling (η_t,ξ_t) of zero-range processes with initial states η and ξ, respectively, so that $\eta_t\leqslant\xi_t$ for all $t\geqslant 0$. (We refer to Ref. 1 for the details, but recall one such coupling in the proof of Lemma 3.3.)

Notations 2.2:

(a) On $\mathbb{N}^{\mathbb{Z}}$ we let $(\tau_x)_{x\in\mathbb{Z}}$ denote the group of shifts, given by $\tau_x\eta(y)=\eta(y+x)$, for $\eta\in\mathbb{N}^{\mathbb{Z}}$, x, $y\in\mathbb{Z}$. Also, if μ is a probability measure on $\mathbb{N}^{\mathbb{Z}}$, we let $(\mu\tau_x)(A)=\mu(\tau_x A)$ for any Borel subset A of $\mathbb{N}^{\mathbb{Z}}$.

(b) Let $\mathscr{I}$ be the set of those probability measures on $\mathbb{N}^{\mathbb{Z}}$ that are invariant under (S_t), and let $\mathscr{S}$ be the set of probability measures invariant under $(\tau_x)_{x\in\mathbb{Z}}$.

It is known that under more general assumptions than ours, the set of extremal measures in $\mathscr{I} \cap \mathscr{S}$ can be characterized as a continuous family $\{\nu_\rho : 0 \leqslant \rho < +\infty\}$ of probability measures on $\mathbb{N}^{\mathbb{Z}}$, such that:

(i) ν_ρ is a product measure

(ii) $\rho = \nu_\rho(\eta(0))$

In fact, the ν_ρ are product measures given by

$$\begin{aligned} \nu_\rho(\eta(x) = k) &= \frac{1}{\chi_\varphi} \frac{\varphi^k}{g(1) \cdots g(k)}, \qquad && k \geqslant 1 \\ &= \frac{1}{\chi_\varphi} && k = 0 \end{aligned} \tag{2.3}$$

where $\varphi = \varphi(\rho) \in [0, \lim_{k \to +\infty} g(k))$ and χ_φ is the normalizing factor. [With this notation $\varphi(\rho) = \nu_\rho(g(\eta(0)))$.]

Remark 2.3. When $p(0, 1) + p(0, -1) = 1$ it is true that $\mathscr{I} \subseteq \mathscr{S}$, as proven in Ref. 1. For this problem in a more general context we also refer to Ref. 1. (This fact will not be needed here.)

In this article we are interested in the question of "preservation of local equilibrium" and the "derivation of the hydrodynamic equation." That is, we want to investigate if, for a given family $\{\mu^\varepsilon\}$ of approximate local equilibrium distributions (informally $\mu^\varepsilon \tau_{[x\varepsilon^{-1}]}$ should be close to some $\nu_{\rho(x)}$) we can find a suitable time scaling $T(\varepsilon, t)$ such that the measures $\mu^\varepsilon \tau_{[x\varepsilon^{-1}]} S_{T(\varepsilon, t)}$ are approximately equal to $\nu_{\rho(x,t)}$, where the evolution of $\rho(x, t)$ is given by some known equation. Here we restrict ourselves to $\{\mu^\varepsilon\}$'s that are already product measures for which

$$\mu^\varepsilon(\eta(x) = k) = \nu_{\rho_0(\varepsilon x)}(\eta(0) = k)$$

for some initial profile $\rho_0(\cdot)$.

In order to motivate the hypotheses of our theorems, let us assume, for the moment, that local equilibrium is preserved with $T(\varepsilon, t) = t\varepsilon^{-1}$. Then, a simple computation with the generator L, recalling the form of ν_ρ [given by (2.3)], shows that $\rho(x, t)$ "should" satisfy the equation

$$\frac{\partial \rho}{\partial t}(x, t) + \gamma \frac{\partial}{\partial x} \varphi(\rho(x, t)) = 0 \tag{2.4}$$

where

$$\varphi(\rho) = \nu_\rho(g(\eta(0))) \tag{2.5}$$

according to (2.3). Thus, if $\varphi(\cdot)$ is strictly concave (or convex) in $(0, +\infty)$, this is a genuinely nonlinear conservation law (in the language of Lax[9]) and it presents shock waves for increasing (resp. decreasing) initial data. Having this in mind, it is natural to state the following:

Theorem 2.4. For $0 \leqslant \alpha < \beta < +\infty$, let $\mu_{\alpha,\beta}$ be the product measure such that

$$\mu_{\alpha,\beta}(\eta(x)=k)=\begin{cases}\nu_\alpha(\eta(x)=k) & \text{if } x<0\\ \nu_\beta(\eta(x)=k) & \text{if } x\geqslant 0\end{cases} \tag{2.6}$$

for all $k \geqslant 1$.

Under Assumptions 2.1, and supposing, moreover, that the function $\varphi(\cdot)$ defined by (2.5) is concave, we have

$$\lim_{t\to+\infty} \mu_{\alpha,\beta}\tau_{[vt]}S_t=\begin{cases}\nu_\alpha & \text{if } v<v_c\\ \nu_\beta & \text{if } v>v_c\end{cases}$$

with

$$v_c=\gamma[\varphi(\beta)-\varphi(\alpha)]/(\beta-\alpha) \tag{2.7}$$

(Nothing is said at $v=v_c$!)

Remark 2.5. The function

$$\rho(x, t)=\begin{cases}\alpha & \text{if } x<v_c t\\ \beta & \text{if } x\geqslant v_c t\end{cases}$$

is the Lax solution of (2.4) for $\rho_0(x)=\alpha\mathbb{1}_{(x<0)}+\beta\mathbb{1}_{(x\geqslant 0)}$, and so Theorem 2.4 says that for such an initial profile $\mu_{\alpha,\beta}\tau_{[x\varepsilon^{-1}]}S_{t\varepsilon^{-1}}$ converges to $\nu_{\rho(x,t)}$ for $x/t\neq v_c$, i.e., away from the discontinuity line $[\rho(x, t)=\rho_0(x-v_c t)]$.

Remark 2.6. In the particular case of constant jump rates [i.e., $g(k)=\mathbb{1}_{(k\geqslant 1)}$) the ν_ρ are the product of geometric distributions, and so

$$\varphi(\rho)=\nu_\rho(g(\eta(0)))=\rho(1+\rho)^{-1}$$

so that all the conditions for Theorem 2.4 are satisfied in this case, with $v_c=\gamma(1+\alpha)^{-1}(1+\beta)^{-1}$. At the end of Section 5 we shall investigate simple conditions on $g(\cdot)$ implying the concavity of $\varphi(\rho)$.

A possible generalization of Theorem 2.4 is:

Conjecture 2.7. Let us suppose:

(i) Assumptions 2.1.

(ii) The function $\varphi(\cdot)$ [defined by (2.5)] is strictly concave.

We now take μ^ε to be product measures such that

$$\mu^\varepsilon(\eta(x)=k)=\nu_{\rho_0(\varepsilon x)}(\eta(0)=k)$$

for all $x\in\mathbb{Z}$, $k\geqslant 1$, where $\rho_0(\cdot)$ is increasing, bounded, and piecewise continuous. Letting $\rho(\cdot,\cdot)$ be the "entropy solution" of (2.4) with initial condition $\rho(x,0)=\rho_0(x)$; then:

$$\lim_{\varepsilon\downarrow 0}\mu^\varepsilon\tau_{[x\varepsilon^{-1}]}S_{t\varepsilon^{-1}}=\nu_{\rho(x,t)}$$

for all (x,t), such that x is a continuity point of $\rho(\cdot,t)$.

Remark 2.8. Let us recall (cf. Ref. 17, Chapter 15) that a bounded, measurable function $\rho(x,t)$ is said to be a weak solution of (2.4) if

$$\iint_{\{(x,t):\,t\geqslant 0\}}[\rho\psi_t+\gamma\varphi(\rho)\,\psi_x]\,dx\,dt+\int\rho_0(x)\,\psi(x,0)\,dx=0 \qquad (2.8)$$

for all test functions ψ of class C^1 and compact support on $\mathbb{R}\times[0,+\infty)$. It is easy to check (see, e.g., Ref. 17, p. 248) that (2.8) implies restrictions on the curves of discontinuity of such $\rho(\cdot,\cdot)$; this "jump condition" determines the slope of discontinuity lines. Nevertheless, (2.8) is not strong enough to guarantee uniqueness of "solution." One way to pick an "interesting" solution is by looking at the equation

$$\frac{\partial\rho}{\partial t}+\gamma\varphi'(\rho)\frac{\partial}{\partial x}\rho=\varepsilon\frac{\partial^2\rho}{\partial x^2}$$

with a small noise term and letting $\varepsilon\to 0$. The limit (which under our conditions on φ is unique) will be the "entropy solution." We refer to Ref. 17, Chapters 15 and 16 for these problems and other related definitions.

Remark 2.9. If the initial $\rho_0(\cdot)$ is increasing and continuous and T denotes the time of the appearance of the first discontinuity in $\rho(\cdot,t)$, say (x_0,T), nothing has been proven so far concerning the microscopic state of the zero-range process around the corresponding macroscopic point (x_0,T). This is open even for the case examined in Ref. 18, i.e., $g(k)=\mathbb{1}_{(k\geqslant 1)}$ and $p(x,x+1)=1$, though the conjecture is clear, based on the results of Wick.

We now state the following analogs of Theorem 2.4 and Conjecture 2.7 for the decreasing initial profile. Now (2.4) indicates what to expect, since the classical solution is unique and known.

Theorem 2.10. Under Assumptions (i) and (ii) of Conjecture 2.7, let $\mu_{\alpha,\beta}$ be defined by Eq. (2.6), but now we take $0\leqslant\beta<\alpha<+\infty$. Then

$$\lim_{t\to+\infty}\mu_{\alpha,\beta}\tau_{[vt]}S_t=\nu_{\rho(v)} \qquad (2.9\text{a})$$

where

$$\rho(v)=\begin{cases}\alpha & \text{for} \quad v\leqslant\gamma\varphi'(\alpha)\\ (\varphi')^{-1}(v/\gamma) & \text{for} \quad \gamma\varphi'(\alpha)\leqslant v\leqslant\gamma\varphi'(\beta)\\ \beta & \text{for} \quad v\geqslant\gamma\varphi'(\beta)\end{cases} \tag{2.9b}$$

[$(\varphi')^{-1}$ denotes the inverse function of φ', which is a strictly decreasing C^1 function.]

Theorem 2.11. Under assumptions (i) and (ii) of Conjecture 2.7, let us now take μ^ε as product measures such that

$$\mu^\varepsilon(\eta(x)=k)=v_{\rho_0(\varepsilon x)}(\eta(0)=k) \tag{2.10}$$

for all $x\in\mathbb{Z}$ and $k\geqslant 1$, where $\rho_0(\cdot)$ is a strictly decreasing, C^1, and bounded function of $\mathbb{R}$ into $[0, +\infty)$. Then, for each $x\in\mathbb{R}$, $t\geqslant 0$:

$$\lim_{\varepsilon\downarrow 0}\mu^\varepsilon\tau_{[x\varepsilon^{-1}]}S_{t\varepsilon^{-1}}=v_{\rho(x,t)} \tag{2.11}$$

where $\rho(\cdot,\cdot)$ is the unique classical solution of (2.4) with initial condition $\rho(x,0)=\rho_0(x)$.

Throughout the next section we suppose:

(A) Assumptions 2.1.

(B) The function $\rho \rightsquigarrow \varphi(\rho) =_{\text{def}} v_\rho(g(\eta(0)))$ is concave.

(When necessary we shall use the smoothness of φ; this follows easily from the definitions.)

3. PROOF OF THEOREM 2.4

The main point of the proof is to exploit the "attractiveness" of the model, and it will be done in several steps.

Lemma 3.1. Let μ be a probability measure on $\mathbb{N}^{\mathbb{Z}}$ such that

(i) $v_\alpha\leqslant\mu\leqslant v_\beta$ for some $0\leqslant\alpha<\beta<+\infty$.

(ii) Either $\mu\tau_1\leqslant\mu$ or $\mu\tau_1\geqslant\mu$.

Then, any sequence $T_n\uparrow+\infty$ has a subsequence T_{n_k} for which there exists D dense (countable) subset of $\mathbb{R}$ such that for each $v\in D$

$$\lim_{k\to+\infty}\frac{1}{T_{n_k}}\int_0^{T_{n_k}}\mu\tau_{[vt]}S_t\,dt=\mu_v \tag{3.1}$$

for some $\mu_v\in\mathcal{I}\cap\mathcal{S}$.

Proof. First, let us make two simple but useful remarks:

1. Since $\mu \leqslant \nu_\beta$, we have $(1/T)\int_0^T \mu\tau_{[vt]}S_t\,dt \leqslant \nu_\beta$ for all $T>0$, $v\in\mathbb{R}$, so that

$$\left\{\frac{1}{T}\int_0^T \mu\tau_{[vt]}S_t\,dt\colon T>0,\, v\in\mathbb{R}\right\}$$

is a precompact set on the space of probability measures on $\mathbb{N}^{\mathbb{Z}}$ (w^*-topology).

2. From Assumption 2.1(b) we see that τ_x and S_t commute (viewed as operators on the space of probability measures on $\mathbb{N}^{\mathbb{Z}}$).

Let us assume (i) and (ii) above, and let $T_n \nearrow +\infty$. To fix ideas, we assume $\mu\tau_1 \geqslant \mu$ in (ii), the other case being analogous. Letting A be a countable dense subset of $\mathbb{R}$, a diagonal argument provides subsequence (T_{n_k}) so that

$$\frac{1}{T_{n_k}}\int_0^{T_{n_k}} \mu S_t\tau_{[vt]}\,dt$$

converges as $k\to+\infty$, provided $v\in A$. We call μ_v this limit, and let $(v\in A)$

$$\rho(v)=\mu_v(\eta(0)) \tag{3.2a}$$

$$L(v)=\lim_{n\nearrow+\infty}\mu_v\tau_n(\eta(0)) \tag{3.2b}$$

$$l(v)=\lim_{n\downarrow-\infty}\mu_v\tau_n(\eta(0)) \tag{3.2c}$$

Since the sequences in (3.2b) and (3.2c) are monotone and bounded, both limits exist, and $l(\cdot)\leqslant\rho(\cdot)\leqslant L(\cdot)$ on A. Also, $\rho(\cdot)$ is increasing in A, and so we can find a countable, dense set D such that $\rho(\cdot)$ has a unique increasing extension to $A\cup D$, still denoted by $\rho(\cdot)$, which is continuous at each $v\in D$. Taking a further subsequence, we could assume

$$\lim_k \frac{1}{T_{n_k}}\int_0^{T_{n_k}} \mu\tau_{[vt]}S_t\,dt=\mu_v \tag{3.3}$$

for some probability μ_v, if $v\in A\cup D$, and that (3.2a) holds for $v\in A\cup D$. Extending $l(\cdot)$ and $L(\cdot)$ to $A\cup D$ through the relations (3.2b) and (3.2c), it is easy to see that $u<v$ implies $\rho(u)\leqslant L(u)\leqslant l(v)\leqslant\rho(v)$. Thus, $\rho=l=L$ at the continuity points of $\rho(\cdot)$, in particular, for $v\in D$. Now, (3.3) and $\mu\tau_1\geqslant\mu$ imply that $\mu_v\tau_1\geqslant\mu_v$. Together with $\rho(v)=L(v)$ this gives $\mu_v\in\mathscr{S}$.

Now fix $v \in D$ and let $\bar{S}_t = S_t \tau_{[vt]}$ $(t \geqslant 0)$. Then $\bar{S}_{t+s} = \bar{S}_t \bar{S}_s$ or $\bar{S}_{t+s} = \bar{S}_t \bar{S}_s \tau_1$ and we get (since $\mu\tau_1 \geqslant \mu$)

$$\mu_v \bar{S}_s = \lim_k \frac{1}{T_{n_k}} \int_0^{T_{n_k}} \mu \bar{S}_t \bar{S}_s \, dt$$
$$\leqslant \lim_k \frac{1}{T_{n_k}} \int_0^{T_{n_k}} \mu \bar{S}_{t+s} \, dt = \mu_v$$

and

$$\mu_v \bar{S}_s = \lim_k \frac{1}{T_{n_k}} \int_0^{T_{n_k}} \mu \bar{S}_t \bar{S}_s \, dt$$
$$\geqslant \lim_k \frac{1}{T_{n_k}} \int_0^{T_{n_k}} \mu \tau_{-1} \bar{S}_{t+s} \, dt = \mu_v \tau_{-1} = \mu_v$$

Thus, μ_v is $\bar{S}_t$-invariant. Since it is also translation-invariant, we have $\mu_v \in \mathscr{I}$, concluding the proof. ∎

Lemma 3.2. With the same conditions and notations of Lemma 3.1, and for $v \in D$, we can write $\mu_v = \int \nu_\rho \lambda_v(d\rho)$, where λ_v is a probability on $[\alpha, \beta]$. Also, if $u < v$ are in D,

$$\mu S_{T_{n_k}} \left(\frac{1}{T_{n_k}} \sum_{[uT_{n_k}] \leqslant x \leqslant [vT_{n_k}]} \eta(x) \right) \xrightarrow[k \to +\infty]{} F(v) - F(u) \tag{3.4}$$

with

$$F(w) = w \int \rho \lambda_w(d\rho) - \gamma \int \varphi(\rho) \, \lambda_w(d\rho) \tag{3.5}$$

for $w \in D$.

Proof. The first assertion follows immediately from Lemma 3.1 and the characterization of $\mathscr{I} \cap \mathscr{S}$ previously mentioned (cf. Ref. 1). It remains to prove (3.4). For this we define, if $u < v$ are in D,

$$G(t) = \sum_{x=[ut]+1}^{[vt]} \mu S_t(\eta(x)) + (vt - [vt]) \, \mu S_t(\eta([vt]+1))$$
$$+ ([ut] + 1 - ut) \, \mu S_t(\eta([ut])) \tag{3.6}$$

Then $G(t) = \int_0^t G'(s) \, ds$ and if $ut, vt \notin \mathbb{Z}$,

$$G'(t) = v\mu S_t(\eta([vt]+1)) - u\mu S_t(\eta([ut]))$$
$$+ \sum_{[ut]+1}^{[vt]} \mu S_t(L\eta(x)) + (vt - [vt]) \, \mu S_t(L\eta([vt]+1))$$
$$+ ([ut] + 1 - ut) \, \mu S_t(L\eta([ut])) \tag{3.7}$$

Now, using (3.3), the translation invariance of μ_v, and the expression for $L\eta(x)$ given by (2.1), quite standard calculations yield

$$\frac{G(T_{n_k})}{T_{n_k}} = \frac{1}{T_{n_k}} \int_0^{T_{n_k}} G'(s)\, ds \to F(v) - F(u)$$

if $u < v$ both in D. But the difference between $G(T_{n_k})/T_{n_k}$ and the l.h.s. of (3.4) is bounded by β/T_{n_k}, so that (3.4) follows. The details of this computation, though quite elementary, appear in the Appendix.

Lemma 3.3. With the same notations of Lemma 3.1, if we take $\mu = \mu_{\alpha,\beta}$ defined by (2.6), with $0 \leqslant \alpha < \beta < +\infty$, then

$$\mu_v = \begin{cases} \nu_\alpha & \text{if} \quad v \in D, \quad v < v_c \\ \nu_\beta & \text{if} \quad v \in D, \quad v > v_c \end{cases} \tag{3.8}$$

where v_c is given by Eq. (2.7).

Proof. This will be divided into two parts. First we use a coupling argument (similar to the one employed in Section 2 of Ref. 3) to get the existence of $\underline{v}$, $\bar{v}$ (finite) so that if $v \in D$ and $v > \bar{v}$, then $\lambda_v = \delta_\beta$, while $\lambda_v = \delta_\alpha$ if $v < \underline{v}$. Once we have this information, we shall use the computation with the average number of particles appearing in Lemma 3.2.

To recall the needed coupling, let us first properly couple the initial measures $\mu_{\alpha,\beta}$ and ν_β. To do this, we take η particles distributed according to $\mu_{\alpha,\beta}$ and suitably add ξ particles to the left of the origin, so that $\eta + \xi$ is distributed according to ν_β. Now (η_t, ξ_t) evolves according to the Markov process on $\mathbb{N}^{\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}}$ [$E \times E$ for unbounded rates $g(\cdot)$; cf. Section 5], whose generator $\tilde{L}$ acts on cylinder functions f as

$$\begin{aligned} \tilde{L}f(\eta, \xi) = &\sum_{x,y} g(\eta(x))\, p(x, y)[f(\eta^{x,y}, \xi) - f(\eta, \xi)] \\ &+ \sum_{x,y} [g(\eta(x) + \xi(x)) - g(\eta(x))]\, p(x, y)[f(\eta, \xi^{x,y}) - f(\eta, \xi)] \end{aligned} \tag{3.9}$$

In this situation both (η_t) and $(\eta_t + \xi_t)$ are zero-range processes with initial distributions $\mu_{\alpha,\beta}$ and ν_β, respectively.

We would like to estimate the position of the rightmost ξ particle. For this we make a comparison with another system ζ of particles. First let us label the ξ particles from right to left with superscript indices, using subscript indices for the time. Thus we have $0 \geqslant \xi_0^1 \geqslant \xi_0^2 \geqslant \cdots$ and $\xi_t^1 \geqslant \xi_t^2 \geqslant \cdots$.

The ζ particles, denoted by $\zeta_t^1, \zeta_t^2,\ldots$, will be such that

(a) $\zeta_0^i = \xi_0^i$ for each i.

(b) The ζ particles move independently according to continuous-time random walks whose holding times have rate $c =_{\text{def}} \sup_k [g(k+1) - g(k)]$ and whose transition probabilities are

$$\bar{P}(0, n) = \begin{cases} 0, & n < 0 \\ \sum_{y \leqslant 0} p(0, y), & n = 0 \\ p(0, n), & n > 0 \end{cases}$$

Note that if we have exactly n of the ξ particles at a given site, then we may assume that each of them jumps at a rate $[g(\eta(x)+n) - g(\eta(x))]/n$. But this is at most c, and we also have that the jumps of the ζ particles are stochastically larger than those of ξ particles. So, there is a coupling of ξ and ζ satisfying

$$P(\xi_t^i \leqslant \zeta_t^i) = 1 \qquad \text{for all} \quad t \geqslant 0, \quad \text{all } i \in \mathbb{N}$$

Let now $\bar{\gamma} = \sum_{x>0} x\bar{P}(0, x)$, which is finite by Assumption 2.1(b). Take $\bar{v} = \bar{\gamma} + 2$ and note that the mean number of ζ particles at a site $x < 0$ and at time 0 is $\beta - \alpha$. Applying Wald's Lemma, we obtain

$$E\frac{1}{t} \sum_{x \geqslant [\bar{v}t]} \zeta_t(x) = \frac{\beta - \alpha}{t} \sum_{k \geqslant 0} P(A_t \geqslant [\bar{v}t] + k) \tag{3.10}$$

where $\zeta_t(x)$ is the number of ζ particles at site x and time t, and A_t is the random walk $A_t = \zeta_t^1 - \zeta_0^1$.

But, for t large enough the r.h.s. of (3.10) is bounded above by

$$\begin{aligned} &\frac{\beta - \alpha}{t} \sum_{k \geqslant 0} P\left(\frac{A_t}{t} \geqslant \bar{\gamma} + 1 + \frac{k}{t}\right) \\ &\qquad \leqslant (\beta - \alpha)\frac{[t]+1}{t} \sum_{k \geqslant 1} P\left(\frac{A_t}{t} \geqslant \bar{\gamma} + k\right) \\ &\qquad \leqslant (\beta - \alpha)\frac{[t]+1}{t} E\left|\frac{A_t}{t} - \bar{\gamma}\right| \end{aligned}$$

which will tend to zero as $t \to +\infty$ by the strong law of large numbers and a standard truncation argument. Now, if $v > \bar{v}$ and $k \in \mathbb{Z}$, we get

$$\begin{aligned} &\overline{\lim_{t \to +\infty}}\ E\xi_t([vt] + x) \\ &\qquad \leqslant \overline{\lim_{t \to +\infty}} \frac{1}{[vt] - [\bar{v}t]} \sum_{y = [\bar{v}t]+1}^{[vt]} E\xi_t(y) \\ &\qquad \leqslant \frac{1}{v - \bar{v}} \overline{\lim_{t \to +\infty}} \frac{1}{t} \sum_{y \geqslant [\bar{v}t]} E\zeta_t(y) = 0 \end{aligned} \tag{3.11}$$

Since $(\eta_t+\xi_t)\tau_{[vt]}$ has distribution ν_β, it follows easily that $\lim_{t\to+\infty} Ef(\tau_{[vt]}\eta_t)=\nu_\beta(f)$ for any $v>\bar{v}$ and any f bounded, cylinder function, and so $\mu_v=\nu_\beta$ if $v\in D$ and $v>\bar{v}$. Similarly, we get $\underline{v}$, so that $\mu_v=\nu_\alpha$ for $v\in D$, $v<\underline{v}$.

Next we claim that indeed $\mu_v=\nu_\alpha$ for all $v\in D$, $v<v_c$ [defined by (2.7)]. To see this, let us notice that $\nu_\alpha\leqslant\mu_{\alpha,\beta}\leqslant\nu_\beta$, the attractiveness of the process, and (3.4) imply

$$(v-u)\,\alpha\leqslant v\int\rho\lambda_v(d\rho)-\gamma\int\varphi(\rho)\,\lambda_v(d\rho)-u\int\rho\lambda_u(d\rho)$$

$$+\gamma\int\varphi(\rho)\,\lambda_u(d\rho)\leqslant(v-u)\,\beta \tag{3.12}$$

if $u<v$, both in D. Taking $u<\underline{v}$ and $v<v_c$, both in D, and $u<v$, we have

$$(v-u)\,\alpha\leqslant v\int\rho\lambda_v(d\rho)-\gamma\int\varphi(\rho)\,\lambda_v(dp)-u\alpha+\gamma\varphi(\alpha)$$

i.e.,

$$\gamma\int_{[\alpha,\beta]}[\varphi(\rho)-\varphi(\alpha)]\,\lambda_v(d\rho)\leqslant v\int_{[\alpha,\beta]}(\rho-\alpha)\,\lambda_v(d\rho)$$

But, for $\rho\in[\alpha,\beta]$,

$$\frac{\varphi(\beta)-\varphi(\alpha)}{\beta-\alpha}(\rho-\alpha)\leqslant\varphi(\rho)-\varphi(\alpha)$$

due to the concavity of φ. Thus, we obtain

$$\gamma\frac{\varphi(\beta)-\varphi(\alpha)}{\beta-\alpha}\int_{[\alpha,\beta]}(\rho-\alpha)\,\lambda_v(d\rho)\leqslant v\int_{[\alpha,\beta]}(\rho-\alpha)\,\lambda_v(d\rho)$$

Hence $\int\rho\,\lambda_v(d\rho)=\alpha$ if $v<v_c$, i.e., $\lambda_v=\delta_\alpha$ for such v, proving the claim. {Recall $v_c=\gamma[\varphi(\beta)-\varphi(\alpha)]/(\beta-\alpha)$.} It remains to prove that $\lambda_v=\delta_\beta$ if $v>v_c$ $(v\in D)$. Let $A(k,u,v)$ be the expression on the l.h.s. of (3.4), when $\mu=\mu_{\alpha,\beta}$. If $u,v\in D$ with $u<v_c$, $\bar{v}<v$, from (3.4) and the previous claim we have

$$\lim_{k\to+\infty}A(k,u,v)=v\beta-\gamma\varphi(\beta)-u\alpha+\gamma\varphi(\alpha)=(v-v_c)\,\beta+(v_c-u)\,\alpha \tag{3.13}$$

On the other hand, (3.4) and the monotonicity of $w\to\mu_w$ give

$$\varliminf_k A(k,u',v')\leqslant(v'-u')\int\rho\lambda_{v'}(d\rho)$$

for any $u'<v'$, both in D.

Let us now suppose, by contradiction, that $\lambda_{v'} \neq \delta_\beta$ for some $v_c < v' \leqslant \bar{v}$. Then, fix $\delta > 0$ so that

$$\int \rho \lambda_{v'}(d\rho) \leqslant \beta - \delta$$

and ε, $0 < \varepsilon < v' - v_c$, so that $2\varepsilon\beta \leqslant \frac{1}{2}\delta(v' - v_c)$, and take v, u', u'' in D so that

$$v_c - \varepsilon < u' < v_c < u'' < v_c + \varepsilon, \qquad \bar{v} < v$$

Then

$$\begin{aligned} A(k, u', v) &\leqslant A(k, u', u'') + A(k, u'', v') + A(k, v', v) + O\left(\frac{1}{T_{n_k}}\right) \\ &\leqslant 2\varepsilon\beta + (\beta - \delta)(v' - v_c) + \beta(v - v') + O\left(\frac{1}{T_{n_k}}\right) \end{aligned}$$

and so

$$\lim_k A(k, u', v) \leqslant \beta(v - v_c) - \tfrac{1}{2}\delta(v' - v_c)$$

contradicting (3.13), and thus concluding the proof. ∎

From Lemma 3.3 and the attractiveness of the process we get the following result:

Proposition 3.4. If $0 \leqslant \alpha < \beta < +\infty$, then

$$\lim_{T \nearrow +\infty} \frac{1}{T} \int_0^T \mu_{\alpha,\beta} S_t \tau_{[vt]}\, dt = \begin{cases} \nu_\alpha & \text{if} \quad v < v_c \\ \nu_\beta & \text{if} \quad v > v_c \end{cases} \tag{3.14}$$

where v_c is given by Eq. (2.7). (For $v = v_c$ we do not say anything.)

Proof. Since $\mu_{\alpha,\beta} \leqslant \nu_\beta$ and the process is order-preserving (attractive), the set

$$\left\{ \frac{1}{T} \int_0^T \mu_{\alpha,\beta} S_t \tau_{[vt]}\, dt \colon T > 0,\, v \in \mathbb{R} \right\}$$

is relatively compact. The rest follows easily from Lemma 3.3.

To conclude the proof of Theorem 2.4, we want to replace in Eq. (3.14) the convergence in the Cesaro sense by actual convergence, when $v \neq v_c$. This is contained in the next result:

Proposition 3.5. Let μ be a probability measure on $\mathbb{N}^{\mathbb{Z}}$ such that (a) $\mu \leqslant \nu_\beta$ and (b) $\mu\tau_1 \geqslant \mu$; and assume v_0 finite exists so that

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \mu S_t \tau_{[vt]}\, dt = \nu_\beta$$

for all $v > v_0$. Then

$$\lim_{t\to+\infty} \mu S_t \tau_{[vt]} = \nu_\beta \qquad \text{for all} \quad v > v_0 \tag{3.15}$$

Proof. Since $\mu S_t \tau_{[vt]} \leqslant \nu_\beta$, it will be enough to show that $\lim_{t\to+\infty} \mu S_t \tau_{[vt]}(\eta(x)) = \beta$ for each $v > v_0$ and $x \in \mathbb{Z}$. Now, fix $v > v_0$, $x \in \mathbb{Z}$, and let

$$\delta(t) = \beta - \mu S_t \tau_{[vt]}(\eta(x)) \tag{3.16}$$

so that $0 \leqslant \delta(t)$, and we must show that $\lim_{t\to+\infty} \delta(t) = 0$.

Notice that $\mu S_t \tau_{[vt]}(\eta(x+y))$ increases in y (since $\mu\tau_1 \geqslant \mu$); thus, if we take u so that $v_0 < u < v$, and let

$$f(\eta) \overset{\text{def}}{=} \sum_{y=0}^{[t(v-u)/2]-1} \eta(x-y) \tag{3.17}$$

then

$$\mu S_t \tau_{[vt]}(f) \leqslant \left[\frac{v-u}{2}\, t\right](\beta - \delta(t)) \tag{3.18}$$

Now, from the expression

$$\mu S_s(f) = \mu(f) + \int_0^s \mu S_r(Lf)\, dr \tag{3.19}$$

and the inequality

$$\mu S_r(Lf) \leqslant \varphi(\beta) \sum |n|\ p(0, n) = C(\beta) < +\infty \tag{3.20}$$

[this is obtained from (2.1) if we consider only the positive terms in Lf], we get

$$\mu S_{t+s} \tau_{[vt]}(f) \leqslant (\beta - \delta(t)) \left[\frac{v-u}{2}\, t\right] + sC(\beta) \tag{3.21}$$

Therefore, if we take $P \in (0, (v-u)/2 \max(u, 1))$, so that

$$[ut + utP] \leqslant ut(1+P) < \frac{u+v}{2}\, t \leqslant [vt] - \left[\frac{v-u}{2}\, t\right] + 1 \tag{3.22}$$

then (3.17) and (3.21) yield

$$\mu S_{t+s}\tau_{[ut+us]}(\eta(x)) \leqslant \left[\frac{v-u}{2}\,t\right]^{-1} \mu S_{t+s}\tau_{[vt]}(f)$$
$$\leqslant \beta - \delta(t) + C(\beta)\, tP\left[\frac{v-u}{2}\,t\right]^{-1} \tag{3.23}$$

for $0 \leqslant s \leqslant tP$. $\{t$ large enough so that $[\frac{1}{2}(v-u)\,t] \geqslant 1.\}$ It follows that

$$\frac{1}{t(1+P)}\int_0^{t(1+P)} \mu S_r \tau_{[ur]}(\eta(x))\,dr$$
$$\leqslant \beta - \frac{P}{P+1}\,\delta(t) + \frac{P}{P+1}\,\frac{C(\beta)\,tP}{[\frac{1}{2}(v-u)\,t]} \tag{3.24}$$

But the l.h.s. of (3.24) tends to β as $t \to +\infty$, and we get

$$\limsup_{t \to +\infty} \delta(t) \leqslant 2C(\beta)\, P/(v-u)$$

Since P can be taken arbitrarily small, the proposition follows. ∎

Proof of Theorem 2.4. From Propositions 3.4 and 3.5 we have the result for $v > v_c$. The case $v < v_c$ also follows in the same way; "reversing inequalities" in Proposition 3.5, we see that for $v_\alpha \leqslant \mu$ and $\mu\tau_1 \geqslant \mu$, if

$$\lim_T \frac{1}{T}\int_0^T \mu S_t \tau_{[vt]}\,dt = v_\alpha$$

for all $v < v_c$, then we actually have $\lim_t \mu S_t \tau_{[vt]} = v_\alpha$ for $v < v_c$.

4. PROOFS FOR DECREASING INITIAL PROFILE

Proof of Theorem 2.10. We may apply Lemmas 3.1 and 3.2 with $\mu = \mu_{\alpha,\beta}$, where $0 \leqslant \beta < \alpha < +\infty$. If $T_n \nearrow +\infty$ and (T_{n_k}), D, and $\mu_v (v \in D)$ are given by Lemma 3.1, then $\mu_v = \int v_\rho\, \lambda_v(d\rho)$ for λ_v a probability on $[\beta, \alpha]$. We first check the following:

Claim 1. If $v \in D$ and $v < \gamma\varphi'(\alpha)$ $[v > \gamma\varphi'(\beta)]$, then $\lambda_v = \delta_\alpha$ ($\lambda_v = \delta_\beta$, respectively).

The proof of such a claim follows exactly the same argument used to prove Lemma 3.3. Indeed, the coupling argument gives, e.g., that $\lambda_v = \delta_\beta$ provided v is sufficiently large. Now, if $\gamma\varphi'(\beta) < u < v$ and $\lambda_v = \delta_\beta$, then (3.4) and the monotonicity-preserving property (attractiveness) yield

$$(v-u)\,\beta \leqslant v\beta - \gamma\varphi(\beta) - u\int \rho\lambda_u(d\rho) + \gamma\int \varphi(\rho)\,\lambda_u(d\rho)$$

so that

$$u \int_{[\beta,\alpha]} (\rho-\beta)\, \lambda_u(d\rho) \leqslant \gamma \int_{[\beta,\alpha]} [\varphi(\rho)-\varphi(\beta)]\, \lambda_u(d\rho) \tag{4.1}$$

But, on $(\beta, \alpha]$ we do have $[\varphi(\rho)-\varphi(\beta)]/(\rho-\beta) < \varphi'(\beta)$, since φ is strictly concave. It then follows from (4.1) that $\lambda_u = \delta_\beta$ for $u \in D$, $u > \gamma\varphi'(\beta)$. The other case is similar, proving the claim.

From Claim 1 and the attractiveness of the process, just as in Propositions 3.4 and 3.5, we obtain

$$\lim_{t\uparrow+\infty} \mu_{\alpha,\beta}\tau_{[vt]}S_t = \begin{cases} \nu_\alpha & \text{for} \quad v < \gamma\varphi'(\alpha) \\ \nu_\beta & \text{for} \quad v > \gamma\varphi'(\beta) \end{cases} \tag{4.2}$$

Next, we check:

Claim 2. If $v \in D$ and $\gamma\varphi'(\alpha) < v < \gamma\varphi'(\beta)$, then $\lambda_v = \delta_{\rho(v)}$, with $\rho(\cdot)$ defined by (2.9b).

To check such a claim, we first take $\bar{u} < \gamma\varphi'(\alpha) < v < \gamma\varphi'(\beta) < \bar{v}$. From the attractiveness we know

$$\overline{\lim_{t\nearrow+\infty}}\, \mu_{\alpha,\beta}S_t\left(\frac{1}{t}\sum_{[\bar{u}t]\leqslant x\leqslant[vt]} \eta(x)\right) \leqslant (v-\bar{u})\,\alpha \tag{4.3}$$

Also, by Eq. (3.4) and Claim 1

$$\lim_{t\nearrow+\infty} \mu_{\alpha,\beta}S_t\left(\frac{1}{t}\sum_{[\bar{u}t]\leqslant x\leqslant[\bar{v}t]} \eta(x)\right) = \bar{v}\beta - \gamma\varphi(\beta) - \bar{u}\alpha + \gamma\varphi(\alpha) \tag{4.4}$$

Thus, from (4.3) and (4.4), it follows that

$$\underline{\lim}_{t\nearrow+\infty}\, \mu_{\alpha,\beta}S_t\left(\frac{1}{t}\sum_{[vt]\leqslant x\leqslant[\bar{v}t]} \eta(x)\right) \geqslant \bar{v}\beta - \gamma\varphi(\beta) + \gamma\varphi(\alpha) - v\alpha \tag{4.5}$$

Since $\gamma\varphi'(\alpha) < v < \gamma\varphi'(\beta)$, we can write $v = \gamma\varphi'(\theta)$, where $\beta < \theta < \alpha$ and $\rho(v) = \theta$, according to (2.9b). Taking any θ' such that $\theta < \theta' < \alpha$, we can use (4.5) for $\mu_{\theta',\beta}$, the fact that $\mu_{\theta',\beta} \leqslant \mu_{\alpha,\beta}$, and the attractiveness to get

$$\underline{\lim}_{t\nearrow+\infty}\, \mu_{\alpha,\beta}S_t\left(\frac{1}{t}\sum_{[vt]\leqslant x\leqslant[\bar{v}t]} \eta(x)\right) \geqslant \bar{v}\beta - \gamma\varphi(\beta) + \gamma\varphi(\theta') - v\theta' \tag{4.6}$$

On the other hand, when $v \in D$ we also know from (3.4) that the l.h.s. of (4.6) is at most

$$\bar{v}\beta - \gamma\varphi(\beta) + \gamma \int \varphi(\rho)\, \lambda_v(d\rho) - v \int \rho\, \lambda_v(d\rho)$$

Thus, taking limits as $\theta' \downarrow \theta$, we get

$$\int_{[\alpha,\beta]} [\gamma\varphi(\rho) - v\rho]\, \lambda_v(d\rho) \geqslant \gamma\varphi(\theta) - v\theta \tag{4.7}$$

But the strict concavity of $\varphi(\cdot)$ implies

$$\max_{\alpha \leqslant \rho \leqslant \beta} [\gamma\varphi(\rho) - v\rho] = \gamma\varphi(\theta) - v\theta$$

and the maximum is attained *only* at $\theta = (\varphi')^{-1}(v/\gamma)$, provided $\gamma\varphi'(\alpha) < v < \gamma\varphi'(\beta)$. The claim thus follows from (4.7).

Since the measures $(1/T)\int_0^T \mu_{\alpha,\beta}\tau_{[vt]}S_t\,dt$ depend monotonically on v and form a relatively compact set, it easily follows that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mu_{\alpha,\beta}\tau_{[vt]}S_t\,dt = v_{\rho(v)} \tag{4.8}$$

for all v. It remains to prove that $\mu_{\alpha,\beta}\tau_{[vt]}S_t \to v_{\rho(v)}$ as $t \to +\infty$, when $\gamma\varphi'(\alpha) < v < \gamma\varphi'(\beta)$. [From continuity of $\rho(\cdot)$ the result will then follow also at $v = \gamma\varphi'(\alpha)$ or $v = \gamma\varphi'(\beta)$.] Now we shall check that if $\tilde{\mu}_v$ is a weak limit point of $\mu_{\alpha,\beta}\tau_{[vt]}S_t$, then we should have:

$$\begin{aligned} &\text{(a)} \quad \tilde{\mu}_v \geqslant v_{\rho(v)} \\ &\text{(b)} \quad \tilde{\mu}_v(\eta(0)) = \rho(v) \end{aligned} \tag{4.9}$$

Indeed, given $\gamma\varphi'(\alpha) < v < \gamma\varphi'(\beta)$, let $\theta = \rho(v)$, so that $\beta < \theta < \alpha$ and $\gamma\varphi'(\theta) = v$. If $\tilde{\theta}$ is such that $\beta < \tilde{\theta} < \theta < \alpha$, then

$$\mu_{\tilde{\theta},\beta}\tau_{[vt]}S_t \leqslant \mu_{\alpha,\beta}\tau_{[vt]}S_t \tag{4.10}$$

and $v < \gamma\varphi'(\tilde{\theta})$, so that (4.2) implies

$$\lim_{t\to+\infty} \mu_{\tilde{\theta},\beta}\tau_{[vt]}S_t = v_{\tilde{\theta}} \tag{4.11}$$

Part (a) in Eq. (4.9) now follows from (4.10) and (4.11). To get (b), we notice that

$$\int_{\bar{u}}^{\bar{v}} \rho(w)\,dw = \bar{v}\beta - \gamma\varphi(\beta) - \bar{u}\alpha + \gamma\varphi(\alpha)$$

for $\bar{u} < \gamma\varphi'(\alpha) < \gamma\varphi'(\beta) < \bar{v}$, so that, from (3.4),

$$\lim_{t\nearrow+\infty} \mu_{\alpha,\beta}S_t\left(\frac{1}{t}\sum_{[\bar{u}t]\leqslant x\leqslant[\bar{v}t]} \eta(x)\right) = \int_{\bar{u}}^{\bar{v}} \rho(w)\,dw \tag{4.12}$$

and (b) follows from (4.12), and (a) in (4.9) with the same argument as in Ref. 3, p. 332 (proof of Theorem 3.2), but reversing inequalities. This completes the proof. ∎

We conclude this section showing how to use the result just proven to get Theorem 2.11. Basically, we use the method of characteristics for (2.4), and the attractiveness of these processes. In Ref. 3 this argument was partially used in the proof of the same result when $p(x, x+1)=1$ and $g(k)=\mathbb{1}_{(k \geqslant 1)}$. Nevertheless, they also use some particularities of the coupling, which are not possible in this more general context. Assuming that $\rho_0(\cdot)$ is bounded, these can be avoided, as we show next.

Proof of Theorem 2.11. For $\rho_0(\cdot)$ satisfying our hypotheses, the unique smooth solution of (2.4) can be obtained through the method of characteristics. For any given pair (x, t) with $t>0$, the characteristics through (x, t) intersect $\mathbb{R} \times \{0\}$ at a unique point $(y(x, t), 0)$ and

$$x - y(x, t) = t\gamma\varphi'(\rho_0(y(x, t)))$$

so that

$$\rho(x, t) = \rho_0(x - t\gamma\varphi'(\rho_0(y(x, t))))$$

is the solution of (2.4) (Ref. 17, p. 243). Fix (x, t) and let $\bar{\rho}$ be such that $\rho(x, t) < \bar{\rho} < \sup_y \rho_0(y)$. There exists a unique z such that $\bar{\rho} = \rho_0(z)$, and we have $z < y(x, t)$. Also, the strict concavity of $\varphi(\cdot)$ implies that $\varphi'(\rho_0(z)) < \varphi'(\rho_0(y(x, t)))$. Thus

$$x - z > t\gamma\varphi'(\rho_0(z)) \tag{4.13}$$

Let μ_z^ε be the product measure such that

$$\begin{aligned} \mu_z^\varepsilon(\eta(u)=k) &= \nu_{\rho_0(z)}(\eta(u)=k) \qquad \text{if} \quad u \geqslant [z\varepsilon^{-1}]+1 \\ &= \nu_{\rho^*}(\eta(u)=k) \qquad \text{if} \quad u \leqslant [z\varepsilon^{-1}] \end{aligned}$$

where $\rho^* = \sup_r \rho_0(r) \in [0, +\infty)$. Thus,

$$\mu^\varepsilon \leqslant \mu_z^\varepsilon = \mu_{\rho^*, \rho_0(z)} \tau_{-[z\varepsilon^{-1}]-1}$$

and by the attractiveness

$$\mu^\varepsilon \tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \leqslant \mu_{\rho^*, \rho_0(z)} \tau_{[x\varepsilon^{-1}]-[z\varepsilon^{-1}]-1} S_{t\varepsilon^{-1}} \tag{4.14}$$

From (4.13), (4.14), and (2.9) it follows that if $\tilde{\mu}$ is a weak limit point of $\mu^\varepsilon \tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}}$, then $\tilde{\mu} \leqslant \nu_{\rho_0(z)}$. Since $\rho_0(z) = \bar{\rho}$ can be taken arbitrarily close $\rho(x, t)$, we see that $\tilde{\mu} \leqslant \nu_{\rho(x, t)}$.

Conversely, letting $\inf_y \rho_0(y) < \rho_- < \rho(x, t)$, we have $\rho_- = \rho_0(y)$ for some $y > y(x, t)$ and $x - y < t\varphi'(\rho_0(y))$. Letting now $\tilde{\mu}_y^\varepsilon = \mu_{\rho(y), 0} \tau_{-[y\varepsilon^{-1}]-1}$ and arguing as before, we see that if $\tilde{\mu}$ is a weak limit point of

$\mu^\varepsilon \tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}}$, then $\nu_{\rho(x,t)} \leqslant \tilde{\mu}$. On the other hand, from the boundedness of $\rho_0(\cdot)$, we already know that $\{\mu^\varepsilon \tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}}: \varepsilon \in (0, 1)\}$ is a relatively compact set, and the proof follows. ∎

5. EXTENSIONS AND COMPLEMENTS

Remark 5.1. For simplicity we have stated the results under the assumption of a bounded rate function $g(\cdot)$. Nevertheless, this was *not* really used in the proofs. That is, if we make the following assumptions:

Assumption 5.2.

(a) $g: \mathbb{N} \to [0, +\infty)$ is nondecreasing, $g(0) = 0 < g(1)$, and $\sup_k [g(k+1) - g(k)] < +\infty$.

(b) Same as Assumption 2.1(b).

Then exactly the same results as stated in Section 2 continue to hold if we relax Assumptions 2.1 to Assumptions 5.2.

Indeed, the only point is the needed modification for the construction of the process itself. Now, one has to restrict the set E of allowed configurations, but it is possible to construct (S_t) strongly continuous, Markov semigroup corresponding to the pregenerator L given by (2.1). Using the construction of Ref. 1, we see that under Assumptions 5.2 the measures ν_ρ defined by (2.3) do satisfy $\nu_\rho(E) = 1$, and $\{\nu_\rho: 0 \leqslant \rho < +\infty\}$ is the set of extremal measures in $\mathscr{I} \cap \mathscr{S}$. Now, the reader can immediately check that all arguments previously used do apply to this situation. The crucial point is the first statement of Lemma 3.3, but the necessary boundedness is contained in (3.10), corresponding exactly to what we do have.

Remarks 5.3. Cocozza[5] studied a class of interacting particle systems so that the rate at which a particle at site x jumps to site y is $b(\eta(x), \eta(y))\, p(x, y)$, where the jump probabilities $p(\cdot, \cdot)$ are translation-invariant and the rate function $b(\eta(x), \eta(y))$ increases in $\eta(x)$ and decreases in $\eta(y)$. Such a condition on $b(\cdot, \cdot)$ yields the property of attractiveness for this class of processes (called "misanthropes"). The point of this remark is to notice that the arguments of Sections 3 and 4 can be extended to this situation. More precisely, let us make the following:

Assumptions 5.4:

(a) $b: \mathbb{N} \times \mathbb{N} \to [0, +\infty)$ is bounded, $b(0, \cdot) \equiv 0$, $b(1, j) > 0$ for all j; $i \to b(i, j)$ is an increasing function for each j; $j \to b(i, j)$ is a decreasing function for each i.

(b) Assumptions (2.3) and (2.4b) of Ref. 5.

(c) $p(x, y)$ satisfies Assumptions 2.1(b) of the present article.

The generator L of the process acts on cylinder functions f on $\mathbb{N}^{\mathbb{Z}}$ as

$$Lf(\eta) = \sum_{x,y} b(\eta(x), \eta(y))\, p(x, y)[f(\eta^{x,y}) - f(\eta)] \tag{5.1}$$

and for any $\eta \in \mathbb{N}^{\mathbb{Z}}$ there exists a unique probability P_η on $D([0, +\infty), \mathbb{N}^{\mathbb{Z}})$ that solves the martingale problem associated to such L (Theorem 1.3 of Ref. 5). Let (S_t) be the corresponding Markov semigroup, and let $\mathscr{I}$ be the set of (S_t)-invariant probability measures.

Under hypotheses more general than Assumptions 5.4, it is proven in Ref. 5 that the extremal measures of $\mathscr{I} \cap \mathscr{S}$ form a one-parameter family $(\nu_\rho : 0 \leqslant \rho < +\infty)$, characterized by:

(i) $\nu_\rho(\eta(x)) = \rho$

(ii) ν_ρ is a product measure and

$$\frac{\nu_\rho(\eta(x) = i+1)}{\nu_\rho(\eta(x) = i)} = \frac{\nu_\rho(\eta(x) = 1)}{\nu_\rho(\eta(x) = 0)} \frac{b(1, i)}{b(i+1, 0)} \tag{5.2}$$

which extends (2.3).

If we set

$$h(\rho) = \sum_y y p(0, y)\, \nu_\rho(h(\eta(0), \eta(y))) \tag{5.3}$$

it is easily seen that now the "hydrodynamic equation" should be

$$\begin{gathered} \frac{\partial \rho}{\partial t}(x, t) + \frac{\partial}{\partial x} h(\rho(x, t)) = 0 \\ \rho(\cdot, 0) = \rho_0(\cdot) \end{gathered} \tag{5.4}$$

where $\rho_0(\cdot)$ is the initial (macroscopic) profile.

What we can easily get from the previous work is that by changing $\gamma\varphi(\cdot)$ to $h(\cdot)$ everywhere, than all the results of Section 2 do extend to this situation. That is, if we assume:

(i) Assumptions (5.4)

(ii) The function $h(\cdot)$ is concave

then:

A. Theorem 2.4 is true, provided we take $v_c = [h(\beta) - h(\alpha)]/(\beta - \alpha)$.

B. Assuming, moreover, that $h(\cdot)$ is smooth (C^2 suffices) we expect Conjecture 2.7 to hold in this case, too, provided Eq. (2.4) is replaced by Eq. (5.4).

C. Similarly, changing $\gamma\varphi(\cdot)$ to $h(\cdot)$, Theorems 2.10 and 2.11 continue to hold.

We shall omit the details of the proofs, since they are essentially the same as in the case of zero range. The points to be careful about are:

1. The computation of Lemma 3.2 (cf. Appendix), which works just as before if we take

$$F(w) = w \int \rho \, \lambda_w(d\rho) - \int h(\rho) \, \lambda_w(d\rho)$$

2. The coupling on the first part of Lemma 3.3.

Note that if b is given by

$$b(i, j) = \begin{cases} 0 & \text{if } j \geqslant 1 \\ i \wedge 1 & \text{if } j = 0 \end{cases}$$

then restricting the process to $\{0, 1\}^{\mathbb{Z}}$, we obtain the simple exclusion process, and $h(\rho) = \gamma\rho(1-\rho)$. Since this is concave, our proofs apply to this case, and thus the results of Refs. 4, 11, 12, and 16 are extended. [Of course, the above rate function $b(\cdot, \cdot)$ does not satisfy Assumption 5.4(a), as required in Ref. 5, but this does not pose any problem for our proofs, since the characterization of $\mathscr{I} \cap \mathscr{S}$ in this case is known.]

APPENDIX

Here we present the computation leading to (3.4). Letting $G(\cdot)$ be defined by (3.6), we have

$$\frac{G(T_{n_k})}{T_{n_k}} = \frac{1}{T_{n_k}} \int_0^{T_{n_k}} G'(s) \, ds \tag{A.1}$$

and if $ut, vt \notin \mathbb{Z}$, $G'(t)$ is given by Eq. (3.7). From (3.1), the translation invariance of μ_v, and Assumption 2.1(b), we see that

$$\frac{1}{T_{n_k}} \int_0^{T_{n_k}} G'(t) \, dt - [v\mu_v(\eta(0)) - u\mu_u(\eta(0))] - \frac{1}{T_{n_k}} \int_0^{T_{n_k}} I_{u,v}(t) \, dt \tag{A.2}$$

tends to zero as $k \to \infty$, where

$$\begin{aligned}
I_{u,v}(t) = & - \sum_{x \leqslant [vt]} \mu S_t(g(\eta(x)))\{P(Y > [vt] + 1 - x) \\
& + (1 - vt + [vt])\, P(Y = [vt] + 1 - x)\} \\
& + \sum_{x > [vt] + 1} \mu S_t(g(\eta(x)))\{P(Y \leqslant [vt] - x) \\
& + (vt - [vt])\, P(Y = [vt] + 1 - x) \\
& - \sum_{[ut] + 1 \leqslant x} \mu S_t(g(\eta(x)))\{P(Y < [ut] - x) \\
& + (ut - [ut])\, P(Y = [ut] - x)\} \\
& + \sum_{x < [ut]} \mu S_t(g(\eta(x)))\{P(Y > [ut] + 1 - x) \\
& + ([ut] + 1 - ut)\, P(Y = [ut] + 1 - x)\} \\
& + \mu S_t(g(\eta([vt] + 1)))(1 - vt + [vt])\, P(Y \leqslant -1) \\
& + \mu S_t(g(\eta([ut])))(ut - [ut])\, P(Y \geqslant 1) \\
& - \mu S_t(g(\eta([vt] + 1)))(vt - [vt])\, P(Y \geqslant 1) \\
& - \mu S_t(g(\eta([ut])))([ut] + 1 - ut)\, P(Y \leqslant -1)
\end{aligned}$$

where Y is a random variable such that

$$P(Y = y) = p(0, y)$$

for $y \in \mathbb{Z}$. Again, using (3.1) we get

$$\begin{aligned}
& \lim_{k \uparrow +\infty} \frac{1}{T_{n_k}} \int_0^{T_{n_k}} I_{u,v}(t)\, dt \\
& = -\mu_v(g(\eta(0))) \left\{ \sum_{z \geqslant 1} P(Y \geqslant z) - \sum_{z \leqslant -1} P(Y \leqslant z) \right\} \\
& \quad + \mu_u(g(\eta(0))) \left\{ \sum_{z \geqslant 1} P(Y \geqslant z) - \sum_{z \leqslant -1} P(Y \leqslant z) \right\} \\
& = \gamma[\mu_u(g(\eta(0))) - \mu_v(g(\eta(0)))]
\end{aligned}$$

proving the Lemma.

Remark. In (A.2) we use $\sum_y |y|\, p(0, y) < +\infty$ to add terms such as $\sum_{x \leqslant [ut]} P(Y > [vt] - x)$ and $\sum_{x > [vt]} P(Y \leqslant [ut] - x)$ to the exact expression of $L(\cdot)$. Now if $u < v$, as $t \uparrow +\infty$, these tend to zero by the dominated convergence theorem if $E\,|Y| < +\infty$.

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